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A note on the Atiyah–Singer index theorem for manifolds with totally antisymmetric H torsion

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Abstract. We present two proofs of the Atiyah-Singer index theorem for the Dirac operator for even-dimensional orientable manifolds without boundary with totally antisymmetric torsion with vanishing curl (*H torsion*) which, for example, may be provided by the field strength of the antisymmetric tensor occurring in the supergravity multiplet of superstring-inspired ten-dimensional theories, or by the structure constants of the group manifold in Kaluza-Klein theories. One of the methods, which is considered as *formal*, utilises a modified version of the conventional heat kernel regularisation. The other which is shown to be connected to the first one through a Legendre transformation uses the supersymmetric path integral approach. Of course, as far as the index is concerned, the H parts form a globally defined exact form and hence our result is consistent with the invariance of the index under the inclusion of torsion. The obvious advantage of our approach is the simplicity of the calculations in contrast to conventional regularisation methods which are practically impossible to handle beyond four dimensions, despite the trivial effects of the torsion on the (physical) results.

A torsion interpretation of the field strength of the antisymmetric tensor $B_{\mu\nu}$ occurring in the supergravity multiplet of the Chapline-Manton-Green-Schwarz ten-dimensional theory (Chamseddine 1981, Bergshoeff *et al* 1982, Chapline and Manton 1983, Green and Schwarz 1984) arises naturally within the context of a low-energy analysis of the heterotic string theory (Daniel and Mavromatos 1986, Sen 1986, Hull and Townsend 1986, Cai and Núñez 1987, Bento and Mavromatos 1987, Gross and Sloan 1987, Mavromatos 1987). As a consequence, the existence of H torsion parts in the Lorentz Chern-Simons modification of the field strength $\mathrm{d}B$ of the antisymmetric tensor by Green and Schwarz (1984) brings up the interesting question of which effects these terms have on the anomaly cancellation mechanism of the ten-dimensional theory. The most elegant (although *formal*) way of studying these effects is the index theorem approach. According to this, the ten-dimensional non-Abelian gauge and gravitational anomalies are determined by the chiral anomalies of Dirac or Rarita-Schwinger operators in a spacetime with two dimensions higher. The latter are merely the index density (up to irrelevant globally defined total divergence terms) of the pertinent Atiyah-Singer index theorem (Atiyah and Singer 1968). In this way it has been shown (Hull 1986) that the H parts of the Lorentz Chern-Simons terms act like local counterterms in the effective action and therefore have no effect on the cancellation mechanism. Of course, as has been shown by Chern (1979), the index theorem is unaffected by torsion terms since they form globally defined exact forms and thereby

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drop out under integration over the compact $2d$-dimensional orientable manifold M without boundary which simulates spacetime. Hence, the result concerning the torsion parts of the chiral anomaly depends on the prescription one uses for the computation. In spite of the trivial role of the torsion, however, in the physical literature the problem of studying the chiral anomalies in torsionful spaces has been confined, so far, to four-dimensional Einstein-Cartan spaces (Obukhov 1983, Yazima and Kimura 1986, Barth 1987). The reason is that the heat kernel methods used for the regularisation of the infinite trace of the chirality matrix Γ_{d+1}, which practically determines the chiral anomaly, are highly dimension dependent and are so complicated when one tries to go beyond four dimensions that any attempt of extracting results without a computer seems impossible. Hence, the heat kernel methods, although probably the most rigorous approach to the problem, must be abandoned, unless new techniques are found. It is the purpose of this note to show that a modified version of the heat kernel, based on a background expansion, yields results amazingly quickly, at least for the case of totally antisymmetric H torsion. In this note we deal with torsion provided by the antisymmetric tensor field strength. We stress, however, that the method is *formal* to some extent as we shall see below. Its proper justification comes from comparison of the results with more rigorous attempts, like the supersymmetric path integral method to which it is shown to be associated via an appropriate Legendre transformation†.

The structure of this paper is as follows. First we review basic features of the formalism of torsionful spaces, paying particular attention to discussing properties of the H torsion. Next we present a modified heat kernel method which, as we show, simplifies enormously the computation of the relevant Atiyah–Singer index theorem for Dirac (spin-$\frac{1}{2}$) fermions in D dimensions. Comparison of the relevant results for the chiral anomalies (within a given prescription) with those obtained by associated conventional heat kernel approaches is made for the four-dimensional Einstein–Cartan spaces, which constitute the only case where results from conventional heat kernel expansions have been obtained. Connection of this method for the computation of the chiral anomalies with the supersymmetric path integral approach through an appropriate Legendre transformation is explained in some detail.

Throughout this work we shall confine ourselves to the case of the Dirac operator, acting on spin-$\frac{1}{2}$ Dirac fermions. The reason is that if we restrict ourselves only to Lagrangian theories, i.e. theories in which the relevant operator can be derived from an action principle, then the torsion cannot couple consistently to higher-spin fermion states unless we are working with supergravity theories, where, however, the dynamical content of the torsion is quite different since it is determined by the matter fields themselves (Obukhov 1983, de Wit 1984).

In evaluating chiral anomalies for Dirac spin-$\frac{1}{2}$ fermions (which are free from gravitational anomalies) in the presence of totally antisymmetric torsion in the path integral approach we study the object (Fujikawa 1979):

$$\int \mathrm{d}^D x\, \mathscr{A}(x) = \int \mathrm{d}^D x \lim_{N\to\infty} \sum_{n=1}^{N} \varphi_n^{\dagger}(x)\Gamma_{D+1}\varphi_n(x) \tag{1}$$

† To our knowledge, a similar method has been suggested by Fujikawa *et al* (1986), but the authors restricted themselves to the Riemannian case only. Also we note the work of Mañes and Zumino (1986) where a WKB approximation is used for the evaluation of the heat kernel. The method seems to be related to ours. Again the authors restrict themselves to zero torsion. As we shall discuss below some parts of the relevant calculations in the torsionful case are highly non-trivial, despite the trivial effects of the torsion on the final (physical) results.

where $\varphi_n(x)$ are the orthonormal eigenfunctions of the generalised Dirac operator with totally antisymmetric torsion†

$$\begin{aligned} &\mathrm{i}\mathscr{D}(\omega, H)\psi \equiv \mathrm{i}\tilde{\mathscr{D}}(\tilde{\omega})\psi = \mathrm{i}\Gamma^c e_c^\mu(\partial_\mu - \tfrac{1}{4}\mathrm{i}\tilde{\omega}_\mu^{ab}\sigma^{ab})\psi \\ &\sigma^{ab} = \tfrac{1}{2}\mathrm{i}[\Gamma^a, \Gamma^b]. \end{aligned} \tag{2}$$

The torsion is associated with the antisymmetric part of the affine connection

$$\mathscr{H}^\rho_{\mu\nu} = \tilde{\Gamma}^\rho_{\mu\nu} - \tilde{\Gamma}^\rho_{\nu\mu} \equiv 2\tilde{\Gamma}^\rho_{[\mu\nu]}. \tag{3}$$

The generalised spin connection on the other hand contains the contorsion which is defined by the vielbein postulate

$$\begin{aligned} &\tilde{\mathscr{D}}_\mu(\tilde{\omega}, \tilde{\Gamma})e^a_\nu = e^a_{\nu,\mu} + \tilde{\omega}^a_{b\mu}e^b_\nu - \tilde{\Gamma}^\lambda_{\nu\mu}e^a_\lambda = 0 \\ &\tilde{\mathscr{D}}_\mu(\tilde{\omega}, \tilde{\Gamma})e^\lambda_a = e^\lambda_{a,\mu} - \tilde{\omega}^b_{a\mu}e^\lambda_b + \tilde{\Gamma}^\lambda_{\nu\mu}e^\nu_a = 0. \end{aligned} \tag{4}$$

The contorsion is defined as

$$\tilde{\omega}^{ab}_\mu = \omega^{ab}_\mu + e^a_\rho e^b_\sigma Y^{\rho\sigma}_{\ \ \mu} \tag{5}$$

where ω^{ab}_μ is the torsion-free spin connection.

From (4) it is clear that

$$Y^\rho_{\mu\nu} = \tfrac{1}{2}(\mathscr{H}^\rho{}_{\mu\nu} + \mathscr{H}_{\mu\nu}{}^\rho + \mathscr{H}^\rho{}_{\nu\mu}) \tag{6a}$$

which in the case of totally antisymmetric torsion simplifies to

$$Y^\rho_{\mu\nu} = \tfrac{1}{2}\mathscr{H}^\rho{}_{\mu\nu}. \tag{6b}$$

Also an important consequence of (4) is the covariant constancy of the Γ^i matrices with respect to the torsional covariant derivative

$$[\tilde{\mathscr{D}}_\mu, \Gamma^\rho] = (\tilde{\mathscr{D}}_\mu\Gamma^\rho) = 0. \tag{6c}$$

For the moment we ignore Yang–Mills fields. Their inclusion is straightforward and will be discussed later.

We note that the total antisymmetry of the torsion guarantees that the geodesic equations are unaffected by the inclusion of torsion and that the torsionful covariant derivative is compatible with the metric (i.e. the latter is covariantly constant).

We may regulate (1) in a gauge invariant way by introducing a Gaussian cut-off

$$\int \mathrm{d}^D x\, \mathscr{A}(x) = \int_x \lim_{\beta\to 0} \sum_{n=1}^\infty \varphi_n^\dagger(x)\Gamma_{D+1}\,\mathrm{e}^{-\beta H}\varphi_n(x) \equiv \int_x \lim_{\beta\to 0} \mathrm{Tr}\langle x|\Gamma_{D+1}\,\mathrm{e}^{-\beta H}|x\rangle \tag{7a}$$

where the Hamiltonian operator is the square of the generalised Dirac operator. Due to the total antisymmetry of the H torsion as well as the Bianchi identity:

$$\partial_{[\mu}\mathscr{H}_{\nu\rho\sigma]} = \nabla_{[\mu}\mathscr{H}_{\nu\rho\sigma]} = 0 \tag{7b}$$

which is specific to the H torsion (i.e. derivable from the curl of a two-rank antisymmetric tensor field $B_{\mu\nu}$) we may write this square in the following remarkably simple form:

$$\begin{aligned} \beta H &= -\beta(\Gamma^\mu\tilde{\mathscr{D}}_\mu(\tilde{\omega}))(\Gamma^\nu\tilde{\mathscr{D}}_\nu(\tilde{\omega})) \\ &= -\beta\{g^{\mu\nu}\tilde{\mathscr{D}}_\mu\tilde{\mathscr{D}}_\nu + \tfrac{1}{8}\sigma^{\mu\nu}\sigma^{ab}\tilde{R}_{ab\mu\nu} - \tfrac{1}{2}\mathrm{i}\sigma^{\mu\nu}\mathscr{H}_{\mu\nu}{}^\lambda\tilde{\mathscr{D}}_\lambda\} \\ &= -(\tilde{\mathscr{D}}^\mu - \tfrac{1}{4}\mathrm{i}\mathscr{H}^\mu_{ab}\sigma^{ab})(\tilde{\mathscr{D}}_\mu - \tfrac{1}{4}\mathrm{i}\mathscr{H}_{\mu cd}\sigma^{cd}) + X \\ &\equiv -g^{\mu\nu}\tilde{\mathscr{D}}'_\mu\tilde{\mathscr{D}}'_\nu + X \equiv -\tilde{\Box} + X \end{aligned} \tag{8a}$$

$$\tilde{\mathscr{D}}'_\mu = \partial_\mu - \tfrac{1}{4}\mathrm{i}(\omega + \tfrac{3}{2}H)_\mu{}^{ab}\sigma^{ab} \tag{8b}$$

† Greek characters denote world indices. Latin letters are used for tangent space indices.

where $\tilde{\mathscr{D}}'_\mu$ is a covariant derivative whose spin connection contains three times more contorsion than the corresponding one in the Dirac operator (2).

By X we denote parts without Γ matrices, containing the Riemannian scalar curvature as well as terms of the form H^2, which, as we shall see, do not contribute to the index.

Note that above we made use of special properties of the generalised curvature tensor with torsion. In particular we used the fact that it no longer satisfies the cyclic identity

$$\tilde{R}_{[\mu\nu\rho\sigma]} = -\tfrac{1}{2}\mathscr{H}^{\lambda}_{[\mu\nu}\mathscr{H}_{\rho\sigma]\lambda} \tag{9a}$$

and that the Ricci tensor has an antisymmetric part $\tilde{R}_{\mu\nu}|_A$

$$\tilde{R}_{\mu\nu}|_A = -\tfrac{1}{2}\nabla^\lambda \mathscr{H}_{\lambda\mu\nu} = -\tfrac{1}{2}\tilde{\mathscr{D}}^\lambda(\tilde{\omega}, \tilde{\Gamma})\mathscr{H}_{\lambda\mu\nu}. \tag{9b}$$

Motivated by the background expansion used in the supersymmetric path integral method to obtain the small fluctuation Lagrangian for the relevant one-dimensional supersymmetric system (Alvarez-Gaumé 1983a, b, Mavromatos 1986) we perform the following expansion around a point x_0 (Fujikawa *et al* 1986)

$$x_\mu = x_{0\mu} + \sqrt{\beta}\, y_\mu. \tag{10}$$

For the consistency of the present scheme of course the final results should be independent of y. The scheme (10) is the simplest one (which corresponds to a symmetric ordering prescription in the supersymmetric path integral method as we shall see).

Choosing our coordinate system at x_0 such that (Veltman 1975)

$$\begin{aligned} g_{\mu\nu}(x_0) &= \delta_{\mu\nu} \\ \partial_\alpha g_{\mu\nu}(x_0) &= 0 \end{aligned} \tag{11}$$

and performing appropriate local Lorentz transformations to the fermionic variables we can write ($8a$) in the following form (in the expansion up to $\mathrm{O}(\beta)$ the metric term in $g^{\mu\nu}\tilde{\mathscr{D}}'_\mu\tilde{\mathscr{D}}'_\nu$ contributes only through its flat-space piece, $\delta^{\mu\nu}$, due to the conditions (11):

$$\beta H_0 = -[\partial/\partial y^\mu - \tfrac{1}{4}\mathrm{i}\beta y^\nu \sigma^{ab}\mathbb{R}^{ab}{}_{\nu\mu}[(\omega + \tfrac{3}{2}H)(x_0)]]^2 + \beta X(x_0). \tag{12}$$

Above we used modified Lie transports (Alvarez-Gaumé and Ginsparg 1985), which consist of performing (non-anomalous) local Lorentz transformations to the usual Lie derivative in the expansion of the various quantities in (12). For example, our modified Lie transport for the connection 1-form which is dictated by the necessity of removing the non-covariant terms appearing in the ordinary Lie transport expression is

$$\begin{aligned} &x^\alpha \to x^\alpha - \xi^\alpha(x) \\ &\mathscr{L}'_\xi \omega^{ab}_\mu(x) \equiv \mathscr{L}_\xi \omega^{ab}_\mu(x) + \partial_\mu[-\xi^\gamma \omega^{ab}_\gamma(x)] + [\underline{\omega}_\mu(x), -\xi^\gamma \underline{\omega}_\gamma(x)]^{ab} \\ &\qquad = \xi^\gamma \mathbb{R}^{ab}{}_{\gamma\mu}(x). \end{aligned} \tag{13}$$

Moreover our modified Lie transport will be accompanied by a further local Lorentz rotation which removes the terms proportional to the generalised connection $\tilde{\omega}^{ab}_\mu(x_0)$ in the expansion of ($8a$) around x_0.

An equivalent way of stating this is that we choose our local coordinate system on the tangent space of the manifold based at x_0 in such a way that the generalised connection

$$\tilde{\omega}^{ab}_\mu(x_0) = 0. \tag{14}$$

This can always be achieved in a gauge theory as a consequence of the Schwinger–Fock gauge condition (Fock 1937, Schwinger 1951)

$$\mathscr{A}_\mu(x)(x-x_0)^\mu = 0. \tag{15}$$

The latter is always implementable by means of the transformation (Cronstrom 1980, see also Zuk 1986)

$$\begin{aligned} g(x, x_0) &= P\exp\left(-\int_0^1 \mathrm{d}t(x-x_0)_\mu \mathscr{A}^\mu(x_0+t(x-x_0))\right)g_0(x_0) \\ g_0(x_0) &= g(x_0, x_0) = 1. \end{aligned} \tag{16}$$

By simply requiring this condition to be valid near x_0 we get $\mathscr{A}_\mu(x_0)=0$.

From the above discussion it becomes clear why this method can give *formal* results in the torsionful case. Our basic assumption was that the generalised spin 'connection' transforms as a connection under local Lorentz transformations. This is not the case if the original Riemannian connection is flat, since the contorsion tensor transforms covariantly under local Lorentz transformations as a consequence of the invariance of the torsion tensor and the vielbein rotations. One may argue that we are not interested in such cases since the classical action with fermions and torsion only is not invariant under local Lorentz rotations. However, we understand that arguments of this kind are not convincing to the mathematicians (in fact one can give a counterargument that gauge symmetry does not exist to the free-fermion action in Dirac theory but we do not consider the theory wrong (!)). One way of resolving this difficulty is by restricting ourselves to supergravity theories as in the case of torsion provided by the field strength of the antisymmetric tensor in the supergravity multiplet of the Chapline–Manton–Green–Schwarz theory. In such cases non-trivial gravitational backgrounds are required on account of local supersymmetry. As we mentioned above, however, in such cases the coupling of torsion to higher spin fermion states, like gravitinos, has a quite different dynamical content due to its determination by the matter fields themselves with the result that the torsion cannot be treated as an external gravitational background. In any case, as we shall show below, the justification that this *formal* method yields the correct results comes from its equivalence with the supersymmetric path integral approach (which has been originated by Alvarez-Gaumé and Witten (1984) and established rigorously by Getzler (1984)). Further supporting evidence is provided by the agreement of the results obtained here with the rigorous heat kernel direct computation in four dimensions (Obukhov 1983). Unfortunately as we mentioned above, we cannot perform the heat kernel expansion in higher dimensions due to the extremely complicated tensorial algebra in the torsional case (in the Riemannian case such calculations have been performed by Endo and Takao (1985)).

After this necessary digression we turn our attention to equation ($7a$). Upon using plane-wave decomposition and discarding the terms X which do not have the correct structure to survive the small-β limit, we arrive at the effectively flat problem

$$\begin{aligned} \int_x \mathscr{A}(x) = \int_x \lim_{\beta\to 0} \int \frac{\mathrm{d}^{2d}\kappa}{(2\pi)^{2d}} \exp[-\mathrm{i}\kappa^\mu(x_0+\sqrt{\beta}\,y)_\mu]\Gamma_{D+1} \\ \times \exp(-\beta H)\exp[\mathrm{i}\kappa^\mu(x_0+\sqrt{\beta}\,y)_\mu + \kappa\cdot(\mathrm{O}(\beta))] \end{aligned} \tag{17}$$

where above the plane wave is a shorthand notation for the refined plane-wave expression in curved spacetime which is defined through the modified expression for the δ functional (Fujikawa 1981, deWitt 1975):

$$\delta^{(D)}(x, x') = \frac{1}{\sqrt{g(x')}} \int \frac{d^{2d}\kappa}{(2\pi)^{2d}} \exp(i\kappa^{\mu}\partial_{\mu}\sigma(x, x')) \tag{18a}$$

where $\sigma(x, x')$ is the geodesic biscalar. Equation (17) is derived by expanding

$$\lim_{x \to x'} \mathrm{Tr}[\Gamma_{D+1} \exp(-\beta H(x))\delta^{(D)}(x, x')] \tag{18b}$$

around x_0 (recall that the torsion does not enter the geodesic equation). Upon rescaling $\sqrt{\beta}\ \kappa \to \kappa$ we arrive at the result

$$\begin{aligned} &\int_x \mathscr{A}(x) = \int_y \lim_{\beta \to 0} \frac{1}{\beta^d} \mathrm{Tr}\, \Gamma_{D+1} \exp(-i\kappa y) \exp(-\mathbb{H}_0) \exp(i\kappa y) \\ &\mathbb{H}_0 = (\partial/\partial y^{\mu} - \tfrac{1}{4} i\beta y^{\nu} \tilde{\underline{\mathbb{R}}}_{\nu\mu}(x_0))^2 \qquad \tilde{\underline{\mathbb{R}}}_{\mu\nu} = \sigma^{ab} \tilde{\mathbb{R}}^{ab}{}_{\mu\nu} \end{aligned} \tag{19}$$

which shows that in the limit $\beta \to 0$ only terms with $2d$ Γ matrices and of $O(\beta^d)$ will survive. This was the reason we drop the $X(x_0)$ terms from our computation.

This also leads us to consider $\tilde{\underline{\mathbb{R}}}_{\mu\nu}[(\omega + \frac{3}{2}H)(x_0)]$ as a constant (in y space) commuting electromagnetic field (Fujikawa *et al* 1986). The problem then of evaluating (1) has been reduced to that of a flat-space electromagnetic field which can be evaluated by simply copying the result of Schwinger (1951), upon making the necessary normalisation changes, since we are working in $2d$ dimensions. We note again that the above method yields the integrated chiral anomaly rather than the anomaly itself and therefore can only give information up to explicit total divergences (which are irrelevant so far as the index is concerned anyway). The result then for the index theorem for a generalised Dirac operator with totally antisymmetric torsion is

$$\mathrm{ind}(i\tilde{\mathscr{D}}(\omega + \tfrac{1}{2}H)) = \int_{M^{2d}} \mathrm{Tr}\left[\det\left(\frac{i\hat{\mathbb{R}}/4\pi}{\sinh(i\hat{\mathbb{R}}/4\pi)}\right)\right]\Bigg|_{\mathrm{vol}}. \tag{20a}$$

The determinant refers to the world indices μ, ν of the 'matrix'

$$(\hat{\mathbb{R}})_{\mu}{}^{\nu} \equiv \sigma^{ab} \tilde{\mathbb{R}}^{ab}{}_{\mu}{}^{\nu}(\omega + \tfrac{3}{2}H) \qquad \mu, \nu = 1, \ldots, 2d. \tag{20b}$$

We stress that this is not the curvature 2-form, since in the case of non-zero torsion we have the following property of the generalised curvature

$$\tilde{\mathbb{R}}_{\mu\nu\rho\sigma} - \tilde{\mathbb{R}}_{\rho\sigma\mu\nu} = -\tilde{\mathscr{D}}_{[\nu}\mathscr{H}_{\mu]\rho\sigma} + \tilde{\mathscr{D}}_{[\sigma}\mathscr{H}_{\rho]\mu\nu}. \tag{20c}$$

For the particular case that the torsion satisfies the Bianchi identity ($7b$), the following relation emerges

$$\tilde{\mathbb{R}}_{\mu\nu\rho\sigma}(\omega + \tfrac{1}{2}H) = \tilde{\mathbb{R}}_{\rho\sigma\mu\nu}(\omega - \tfrac{1}{2}H). \tag{20d}$$

Hence the matrix ($20b$) may be interpreted as a generalised matrix-valued curvature two-form but with respect to a connection with torsion $-\frac{3}{2}H$. The standard result for the index is obtained by making the substitution

$$(\hat{\mathbb{R}}(\omega - \tfrac{3}{2}H))_{\mu}{}^{\nu} \to \tfrac{1}{2}\tilde{\mathbb{R}}_{\mu}{}^{\nu}{}_{ab} e^a \wedge e^b. \tag{20e}$$

Thus the result (within the given prescription) may be expressed in terms of the Dirac genus for a manifold with $-\frac{3}{2}H$ torsion. We note at this stage that the torsion-dependent parts of the chiral anomaly are highly prescription dependent. Of course from the work of Chern (1979) and Wu and Zee (1985) we know that the torsion terms conspire to yield globally defined exact forms and, therefore, the index remains unchanged relative to the Riemannian case, as expected from the fact that it is a physical quantity and that as such it should be independent of the prescription used.

Before going into the *formal* connection of this method with the supersymmetric path integral it would be interesting to compare our result (20*a*)-(20*e*) with the four-dimensional direct calculation of Obukhov (1983) and Yazima and Kimura (1986) for Einstein-Cartan spaces. Although above we used the specific form of a (totally antisymmetric) $H = \mathrm{d}B$, we note, however, that in four dimensions only the totally antisymmetric part of the torsion couples to the fermions and the problem is very similar (but not quite) to that of an axial chiral U(1) gauge field in curved space. Hence comparison of the results makes perfect sense.

The result that the direct calculation via heat kernel methods gives for the chiral anomaly in this case is

$$(\mathrm{i}/8\pi^2)[\tfrac{1}{48}\varepsilon^{\alpha\beta\gamma\delta}R_{\mu\nu\alpha\beta}R^{\mu\nu}{}_{\gamma\delta}+\tfrac{1}{96}\varepsilon^{\mu\nu\lambda\rho}f_{\mu\nu}f_{\lambda\rho}$$
$$+(\text{total divergence; irrelevant for our purposes})]. \tag{21}$$

Here $f_{\mu\nu}$ is the field strength of the axial electromagnetic field corresponding to the totally antisymmetric component of the torsion (*axial trace*)

$$\begin{aligned} f_{\mu\nu} &= A_{\mu,\nu} - A_{\nu,\mu} \\ A_\mu &= \tfrac{1}{2}\varepsilon_{\mu\alpha\beta\gamma}\mathscr{H}^{\alpha\beta\gamma}. \end{aligned} \tag{22}$$

On the other hand the result for the topological density (which is obtained by collecting from the expansion of the Dirac genus a (volume) 4-form) is

$$(\mathrm{i}/384\pi^2)\varepsilon^{\alpha\beta\gamma\delta}\tilde{R}_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu}{}_{\gamma\delta}$$
$$=(\mathrm{i}/8\pi^2)[\tfrac{1}{48}\varepsilon^{\alpha\beta\gamma\delta}R_{\mu\nu\alpha\beta}R^{\mu\nu}{}_{\gamma\delta}+\tfrac{1}{864}\varepsilon^{\mu\nu\lambda\rho}f_{\mu\nu}f_{\lambda\rho}$$
$$+\text{total divergence (not the same as in equation (21))}]. \tag{23}$$

Remarkably we observe that (23) corresponds to the result (21) upon substituting

$$A_\mu \to -3A_\mu \tag{24}$$

which is consistent with the result in (20*a*)-(20*e*) (up to total divergence terms which cannot be seen since our approach gives the integrated chiral anomaly). Of course the minus sign in the torsion cannot be seen due to the appearance of the axial potential A_μ in (21) in quadratic expressions.

We now come to a brief discussion on the connection of the present method with the supersymmetric path integral. Since the latter is based on a Lagrangian formalism (Cecotti and Girardello 1982, Alvarez-Gaumé 1983a, b) and by construction its Hamiltonian is the square of the Dirac operator (cf (7)) a Legendre transformation must exist which would connect the two pictures. This transformation can be found by the analogy of our system with the electromagnetic field case. The Legendre transformation assumes the form (Lee 1981, Fujikawa *et al* 1986):

$$\int_y \lim_{\beta\to 0}\mathrm{Tr}\langle y|\Gamma_{D+1}\,\mathrm{e}^{-H_0}|y\rangle$$
$$=\int_{\substack{y\\ \beta\to 0}} N\,\mathrm{Tr}\,\Gamma_{D+1}\int'\prod\mathscr{D}\xi^\mu_{(\tau)}\int\mathscr{D}p^\mu_{(\tau)}\exp\biggl(\int_0^1\mathrm{d}\tau(p_\mu(\tau)\dot{\xi}^\mu_{(\tau)}-H_0(p,\xi))\biggr) \tag{25}$$

(where $\dot{\xi} \equiv \mathrm{d}\xi/\mathrm{d}\tau$) with the periodic boundary conditions on the *bosonic* variables

$$\xi^{\mu}(0) = \xi^{\mu}(1) = y^{\mu}. \tag{26}$$

The Hamiltonian density appearing in (25) is given by H_0 in (19) after the replacement of y^{μ} by $\xi^{\mu}(\tau)$ and $-\mathrm{i}\,\partial/\partial y^{\mu}$ by $p^{\mu}(\tau)$. The integrals over ξ do not include the zero oscillator modes on account of the explicit y integration. This is denoted by a prime.

The normalisation factor N will be fixed by demanding that in the zero-torsion limit the result (25) coincides with the Atiyah–Singer index theorem for the ordinary Dirac operator.

Performing the momentum integrations in (25), by completing the squares, we obtain

$$\int_x A(x) = \int_y \lim_{\beta \to 0} N \operatorname{Tr} \Gamma_{D+1} \int' \left(\prod \mathscr{D}\xi^{\mu}\right) \exp\left(\int_0^1 \mathscr{L}_{\mathrm{E}}(\tau)\, \mathrm{d}\tau\right) \tag{27}$$

where the Lagrangian density $\mathscr{L}_{\mathrm{E}}$ is given by ($\xi \to \sqrt{2}\,\xi$)

$$\mathscr{L}_{\mathrm{E}}(\tau) = \tfrac{1}{2}\dot{\xi}^{\mu}\dot{\xi}_{\mu} + \tfrac{1}{2}\beta\xi^{\nu}\dot{\xi}^{\mu}\tilde{\mathbb{R}}_{\nu\mu}(\omega + \tfrac{3}{2}H)(x_0). \tag{28}$$

This coincides with the small fluctuation (second-order) expansion around constant (in τ) background configurations of the following supersymmetric system (spinning particle in a space with torsion) (Braden 1986, Mavromatos 1986, 1987):

$$\mathscr{L}_{\mathrm{E}} = \tfrac{1}{2}g_{ij}\dot{\phi}^i\dot{\phi}^j - \mathrm{i}g_{ij}\psi^i(\dot{\psi}^j + \tilde{\Gamma}^j_{rs}\psi^r\dot{\phi}^s) \tag{29a}$$

or in tangent space notation

$$\mathscr{L}_{\mathrm{E}} = \tfrac{1}{2}g_{ij}\dot{\phi}^i\dot{\phi}^j - \mathrm{i}\psi^a(\dot{\psi}_a + \tilde{\omega}_{jab}\dot{\phi}^j\psi^b) \tag{29b}$$

after performing the fermionic fluctuation integrations and making the substitutions

$$\begin{aligned} &\psi_0^a \to (\tfrac{1}{2}\Gamma^a) \\ &\prod_a \mathrm{d}\psi_0^a \to (\tfrac{1}{2})^{2d} \operatorname{Tr} \Gamma_{D+1} \end{aligned} \tag{30}$$

which are justified by the commutation relations of the fermionic zero modes (Alvarez-Gaumé 1983a, b).

In (29) the connection $\tilde{\omega} = \omega + \tfrac{1}{2}\gamma H$ contains some torsion whose amount is fixed by the requirement that the quantum Hamiltonian (within a certain prescription) coincides with the square of the operator under consideration. As we shall show below this fixes γ to 3 in the particular prescription we use.

We note at this stage that in the case under study, where the H-torsion is provided by the field strength of the antisymmetric tensor $B_{\mu\nu}$, the system (28) is obtained by dimensional reduction (i.e. dropping spatial derivatives) of a two-dimensional heterotic σ model propagating in graviton and antisymmetric tensor backgrounds. This is not the same system as the one used in Alvarez-Gaumé (1983a, b) where the $N = \frac{1}{2}$ supersymmetry generated by a single supercharge was achieved by equating the two components of the world sheet spinors of the $N = 1$ supersymmetric σ model (di Vecchia and Ferrara 1977, Freedman and Townsend 1981). In the purely Riemannian case the terms in the $N = 1$ model depending on the curvature tensor drop out in such a case due to the Bianchi identity. This is not the case when torsion is present due to (9a). The correct system to be used in our case is the heterotic model in which one uses the chiral projection to set one of the two components of the world sheet spinors to zero (Braden 1986, Mavromatos 1986, 1987).

The system (29) possesses constraints due to the dependence of the fermionic variable momenta on the variables themselves. Its quantisation will therefore proceed via the Dirac bracket formalism (details may be found in Davis *et al* (1984) and extension to torsionful cases in Braden (1986)).

The system (29*a*) has a $N=\frac{1}{2}$ supersymmetry†

$$\begin{aligned}\delta_\varepsilon \phi^i &= \varepsilon \psi^i \\ \delta_\varepsilon \psi^i &= -\tfrac{1}{2}\mathrm{i}\varepsilon\dot{\phi}^i\end{aligned} \tag{31}$$

corresponding to a classical supercharge

$$\begin{aligned}Q_{\mathrm{cl}} &= \psi^i\pi_i - \tfrac{2}{3}\mathrm{i}(\tfrac{1}{2}\gamma H_{irs})\psi^i\psi^r\psi^s \\ \pi_i &= g_{ij}\dot{\phi}^j = \frac{\partial \mathscr{L}_{\mathrm{E}}}{\partial \dot{\phi}^i} + \mathrm{i}g_{lj}\psi^l\tilde{\Gamma}^j_{\ ri}\psi^r.\end{aligned} \tag{32}$$

The quantum supercharge is evaluated using the *symmetric ordering* prescription. Due to the complete antisymmetry of the torsion tensor a symmetrisation of the torsion parts is not necessary (Braden 1986, Mavromatos 1986, 1987)

$$\begin{aligned}Q_{\mathrm{qu}} = -Q^\dagger_{\mathrm{qu}} &= \tfrac{1}{2}(\psi^i\pi_i + \pi_i\psi^i) - \tfrac{1}{3}\mathrm{i}\gamma H_{irs}\psi^i\psi^r\psi^s \\ &= -\mathrm{i}\psi^a[\nabla_a + \tfrac{1}{6}\gamma H_{abc}\psi^b\psi^c] \to -\tfrac{1}{2}\mathrm{i}\Gamma^a\tilde{\mathscr{D}}'_a\end{aligned} \tag{33}$$

where $\tilde{\mathscr{D}}'_a$ is a covariant derivative with torsion $\frac{1}{3}\gamma H$.

In writing the quantum supercharge in the form (33), in which the Christoffel connection parts of the covariant derivative have been separated from their torsion counterparts, we took into account the (implicit) torsion dependence of the operators π^i, as becomes clear from their canonical Dirac brackets which are proportional to the generalised curvature tensors (Braden 1986).

In order that the quantum Hamiltonian of the system coincides with the square of the Dirac operator with torsion H, the torsion in (29) must be $3H$. This is the origin of the factor of three in the amount of torsion in the connection, within the context of the given prescription. We note at this stage that by fixing the prescription for the quantum ordering there are no further ambiguities. In our case the symmetric prescription seems most appropriate in yielding the appropriate supercharge in the Riemannian case. Of course, for the consistency of both calculational schemes the final result must be independent of the prescription used (provided that the prescription is the *correct* one) and in fact it is, since the index is independent of the amount of torsion included in the connection. As we have seen above by adopting the prescription which amounts to the expansion (10), we were led to the supersymmetric path integral result (28) with torsion $3H$ as required by the symmetric prescription for consistency. It is in this sense that we associate the symmetric ordering prescription with the scheme used in the background expansion (10).

The small fluctuation expansion of (29) around constant background configurations can be most easily performed by using normal coordinates (Alvarez-Gaumé *et al* 1982, Fridling and Van de Ven 1986), the passage to which is guaranteed by the general coordinate invariance of the target space and the fact that in the case of $N=\frac{1}{2}$ supersymmetry that we are dealing with there are no constraints on the coordinates

† In the case of group manifolds there is a one-parameter family of such supersymmetries (Braden 1986). For the purposes of this paper, we do not restrict ourselves to this case, which corresponds to the case under study by treating the torsion as covariantly constant.

of the target space which would deserve special attention. The result is (Mavromatos 1986, 1987)

$$\mathcal{L}=\tfrac{1}{2}\dot{\xi}^{\mu}\dot{\xi}_{\mu}+\tfrac{1}{2}\psi_0^a\psi_0^b\tilde{\mathbb{R}}_{ab\nu\mu}(\omega+\tfrac{3}{2}H)(x_0)\xi^{\nu}\dot{\xi}^{\mu}-\mathrm{i}\eta^a\dot{\eta}^a. \tag{34}$$

Under the fermionic fluctuation integration the term $\mathrm{i}\eta^a\dot{\eta}^a$ in (34) does not contribute and we obtain the result (28) upon making the substitutions (30) and the formal correspondence $\mathrm{i}\beta\to\frac{1}{4}$. The latter is justified by the fact that the final result is independent of β (and y).

We stress again that the calculations were specific to the torsion with vanishing curl.

The inclusion of gauge fields is straightforward. In the supersymmetric path integral method they arise by extending the one-dimensional system appropriately. The extension is obtained by dimensionally reducing the gauge sector of the (1, 0) supersymmetric heterotic σ model (Alvarez-Gaumé 1983a, b, Alvarez-Gaumé and Witten 1983, Mavromatos 1987).

From the modified heat kernel method point of view, the Hamiltonian operator now becomes

$$\begin{aligned}\beta H&=-\beta(\Gamma^{\mu}(\partial_{\mu}-\tfrac{1}{4}\mathrm{i}\tilde{\omega}_{\mu}-\mathrm{i}A_{\mu}^{A}T^{A}))^2=-\beta(\Gamma^{\mu}\tilde{\mathscr{D}}_{\mu}(\tilde{\omega}))^2+\tfrac{1}{4}\mathrm{i}[\Gamma^{\mu},\Gamma^{\nu}]F_{\mu\nu}^{A}T^{A}\\ F_{\mu\nu}^{A}&=2\partial_{[\mu}A_{\nu]}^{A}+[A_{\mu},A_{\nu}]^{A}.\end{aligned} \tag{35}$$

The expansion of the Yang-Mills terms in the Dirac operator is obtained by using appropriately gauge rotated Lie transports to write

$$\begin{aligned}\mathscr{L}'_{\xi}\underset{\sim}{A}_{\mu}(x)&=\mathscr{L}_{\xi}\underset{\sim}{A}_{\mu}(x)+\partial_{\mu}[-\xi^{\gamma}\underset{\sim}{A}_{\gamma}(x)]+[\underset{\sim}{A}_{\mu}(x),-\xi^{\gamma}\underset{\sim}{A}_{\gamma}(x)]\\ &=\xi^{\gamma}\underset{\sim}{F}_{\gamma\mu}(x).\end{aligned} \tag{36}$$

Also by a gauge transformation we can get rid of terms proportional to $\underset{\sim}{A}_{\mu}(x_0)$, (which corresponds to the gauge condition (15)). Taking into account that in the small-β limit the commutator of $\frac{1}{2}\beta\sigma^{\mu\nu}\underset{\sim}{F}_{\mu\nu}(x_0)$ with H_0 (cf (15)) is irrelevant for the computation of the index density we obtain the result

$$\begin{aligned}&\int_y\lim_{\beta\to0}\frac{1}{\beta^d}\operatorname{Tr}\langle y|\Gamma_{D+1}\exp[-(H_0+\tfrac{1}{2}\beta\underset{\sim}{F}(x_0))]|y\rangle\\ &\qquad=\int_y\lim_{\beta\to0}\frac{1}{\beta^d}\operatorname{Tr}\langle y|\Gamma_{D+1}\exp(-H_0)\exp(-\tfrac{1}{2}\beta\underset{\sim}{F}(x_0))|y\rangle\end{aligned} \tag{37}$$

which by an appropriate Legendre transformation is related to the aforementioned supersymmetric system small-fluctuation expansion after the fermionic fluctuation integrations.

The result for the relevant index theorem, i.e. integrated chiral anomaly (within the given prescription), may then be expressed in terms of the Chern character

$$\operatorname{ind}(\mathrm{i}\mathscr{D}(\omega+\tfrac{1}{2}H;A))=\int_{M^{2d}}\operatorname{ch}(\mathbb{F})\mathbb{A}(\omega-\tfrac{3}{2}H)\big|_{\mathrm{vol}} \tag{38}$$

where vol indicates that we pick up the volume $2d$-form from the expansion of the integrand, as a result of the Grassmannian integration over the fermion zero modes.

Of course the H parts do not contribute to the index, as we have already mentioned. This can be most easily exhibited by switching on the torsion adiabatically through a homotopy function (Chern 1979, Wu and Zee 1985) and considering the derivative of the Dirac genus (which is defined in terms of generalised Pontryagin numbers) with respect to the homotopy parameter. A crucial property, which makes the rate of change of the Dirac genus be an exact globally defined form, is the Bianchi identity that the generalised curvature 2-forms satisfy,

$$\tilde{D}_t \tilde{\mathbb{R}}_t = 0 \tag{39}$$

where $\tilde{D}_t$ is a covariant derivative 1-form with respect to the connection

$$\tilde{\omega}_t = \omega_0 + \tfrac{1}{2} t H \qquad t \in [0, 1]. \tag{40}$$

Here ω_0 is the torsion-free spin connection.

In this paper we have commented on a simplified method of obtaining the index theorem of the Dirac operator, and evidently the chiral anomaly, for manifolds with totally antisymmetric torsion with vanishing curl. The method consists of using an appropriate background expansion of the Hamiltonian operator in the heat kernel regularisation which reduces the problem to that of a constant commuting electromagnetic field in flat space. The latter has been essentially calculated by Schwinger (in four dimensions, but the extension to D dimensions is straightforward). Although the method appears to be formal, it yields results which are in agreement with rigorous approaches like the supersymmetric path integral to which it is explicitly related by an appropriate Legendre transformation. Moreover in four dimensions it agrees with the rigorous heat kernel expansions. A brief description of the supersymmetric path integral approach to the problem has also been presented. The result is that, within the adopted prescription for the regularisation of the relevant infinite quantities encountered in the computation, the relevant index can be entirely expressed in terms of the Dirac genus of a manifold whose spin connection contains three times more torsion (with the sign reversed) than the operator under consideration (of course, it is understood that the amount of torsion in the connection has no physical significance and it is highly prescription dependent). The property of the H field to appear inside generalised Pontryagin numbers constructed out of the torsional curvature tensor leads to the invariance of the index under any change of the amount of torsion in the connection as naively anticipated from topological or continuity arguments. We must note at this stage that because of the appearance of extra (relative to the Riemannian case) zero modes of the Dirac operator in the presence of torsion (Wu and Zee 1984) any naive continuity arguments about the invariance of the index under the inclusion of torsion are not correct. It is the property of the torsion to appear inside topological invariants, as we have explicitly demonstrated above, that leads to the appearance of the extra zero modes in left–right symmetric pairs.

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